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A Flux-Difference Splitting Method for Steady Euler Equations

E. DICK

Department of Machinery, State University of Ghent,
Sint Pietersnieuwstraat 41, Ghent, Belgium

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The polynomial flux-difference splitting method is applied to steady Euler equations. A discrete set of equations which is both conservative and positive is obtained. The set of equations is solved by the relaxation technique. © 1988 Academic Press, Inc.

1. INTRODUCTION

In recent years, upwind techniques based on flux-vector splitting and flux-difference splitting for solving the Euler equations have gained considerable popularity.

The flux-vector splitting approach was introduced by Steger and Warming [1] for the unsteady Euler equations. This splitting is based on the homogeneity of degree one with respect to the conservative variables ρ, ρu, ρv, ρE. It was shown by Jespersen [2] that the flux-vector splitting approach can also be used directly on the steady Euler equations to generate discrete equations that can be solved by relaxation methods. The technique, however, shows some shortcomings in the treatment of shocks. In the conservative formulation, so-called undifferenced terms appear. These terms represent a loss of positiveness of the discrete set of equations and cause oscillations in the vicinity of shocks.

Going back to the earlier work of Godunov [3] a remedy for the shock oscillations can be found in not splitting the flux-vectors themselves, but the differences of the flux-vectors. Several flux-difference splitting procedures were proposed for unsteady equations, simplifying the Godunov method. The splitting of Roe [4] is based on the homogeneity of degree two of the flux-vectors with respect to the variables $\sqrt{\rho}$, $\sqrt{\rho}\, u$, $\sqrt{\rho}\, v$, $\sqrt{\rho}\, H$. The splitting of Osher [5, 6] is a splitting with respect to the variables $\sqrt{\gamma p/\rho}$, u, v, $\ln(p/\rho^{\gamma})$. An analysis on the relation between Godunov's, Roe's, and Osher's splitting was done by Van Leer [7]. A very simple splitting based on the polynomial character of the flux-vectors with respect to the primitive variables ρ, u, v, p was proposed by Lombard *et al.* [8, 9].

It was shown by Hemker and Spekreijse [10, 11] that the Osher scheme can be used directly on steady Euler equations, leading to a conservative set of discrete

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equations that can be solved by relaxation techniques. Hemker and Spekreijse chose the Osher scheme, although it is the most complex of the mentioned flux-difference splitting schemes, because of its rigour in the construction of the discrete equations and the treatment of the boundary conditions.

In this paper, it is shown that also the simple flux-difference splitting technique of Lombard *et al.*, based on the polynomial character of the flux-vectors can be used directly on the steady Euler equations. In the original approach of Lombard *et al.*, the flux-difference splitting was based on an approximate eigenvector decomposition of Jacobian matrices, leading to possible errors in the vicinity of zero eigenvalues. With the time marching of the time dependent equations, as used by Lombard *et al.*, these errors are not too important. In this paper, the eigenvector decomposition of the Jacobian matrices is done exactly. This results for the steady Euler equations in discrete field- and boundary equations which are completely conservative and positive. This allows relaxation directly on the steady equations. The treatment has the same rigour as the Hemker–Spekreijse approach, however, with a much simpler splitting technique.

2. Polynomial Flux-Difference Splitting

Steady Euler equations, in two dimensions, take the form:

$$\frac{\partial f}{\partial x}+\frac{\partial g}{\partial y}=0, \tag{1}$$

where the flux vectors are

$$f=\begin{pmatrix}\rho u\\ \rho uu+p\\ \rho uv\\ \rho H u\end{pmatrix},\qquad g=\begin{pmatrix}\rho v\\ \rho uv\\ \rho vv+p\\ \rho H v\end{pmatrix}; \tag{2}$$

ρ is density, u and v are Cartesian velocity components, $H=E+p/\rho$ is total enthalpy, p is pressure, $E=p/(\gamma-1)\,\rho+\frac{1}{2}u^2+\frac{1}{2}v^2$ is total energy and γ is adiabatic constant.

Since the components of the flux vectors form polynomials with respect to the primitive variables ρ, u, v, and p, components of flux-differences can be written as

$$\Delta\rho u=\bar{u}\,\Delta\rho+\bar{\rho}\,\Delta u$$

$$\begin{aligned}\Delta\rho uu+p&=\overline{\rho u}\,\Delta u+\bar{u}\,\Delta\rho u+\Delta p\\ &=\bar{u}^2\,\Delta\rho+(\overline{\rho u}+\bar{\rho}\,\bar{u})\Delta u+\Delta p\end{aligned}$$

$$\begin{aligned}\Delta\rho uv&=\overline{\rho u}\,\Delta v+\bar{v}\,\Delta\rho u\\ &=\bar{u}\,\bar{v}\,\Delta\rho+\bar{\rho}\,\bar{v}\,\Delta u+\overline{\rho u}\,\Delta v\end{aligned}$$

$$\Delta\rho H u = \overline{\rho u}(\tfrac{1}{2}\Delta u^2 + \tfrac{1}{2}\Delta v^2) + \tfrac{1}{2}(\overline{u^2} + \overline{v^2})\,\Delta\rho u + \frac{\gamma}{\gamma-1}\,\Delta p u$$

$$= \tfrac{1}{2}(\overline{u^2} + \overline{v^2})\,\bar{u}\,\Delta\rho + \tfrac{1}{2}(\overline{u^2} + \overline{v^2})\,\bar{\rho}\,\Delta u + \overline{\rho u}\,\bar{u}\,\Delta u + \frac{\gamma}{\gamma-1}\,\bar{p}\,\Delta u$$

$$+ \overline{\rho u}\,\bar{v}\,\Delta v + \frac{\gamma}{\gamma-1}\,\bar{u}\,\Delta p,$$

where the bar denotes mean value.

With the definition of

$$\bar{q}^2 = \tfrac{1}{2}(\overline{u^2} + \overline{v^2}) \tag{3}$$

the flux-difference Δf can be written as

$$\Delta f = \begin{pmatrix} \bar{u} & \bar{\rho} & 0 & 0 \\ \bar{u}^2 & \bar{\rho}\,\bar{u} + \overline{\rho u} & 0 & 1 \\ \bar{u}\,\bar{v} & \bar{\rho}\,\bar{v} & \overline{\rho u} & 0 \\ \bar{q}^2\bar{u} & \bar{q}^2\bar{\rho} + \overline{\rho u}\,\bar{u} + \frac{\gamma}{\gamma-1}\,\bar{p} & \overline{\rho u}\,\bar{v} & \frac{\gamma}{\gamma-1}\,\bar{u} \end{pmatrix} \Delta\xi, \tag{4}$$

where $\xi^T = \{\rho, u, v, p\}$.

With the definition of $\bar{\bar{u}}$

$$\bar{\rho}\,\bar{\bar{u}} = \overline{\rho u}, \tag{5}$$

the flux-difference Δf can also be written as

$$\Delta f = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \bar{u} & \bar{\rho} & 0 & 0 \\ \bar{v} & 0 & \bar{\rho} & 0 \\ \bar{q}^2 & \bar{\rho}\,\bar{u} & \bar{\rho}\,\bar{v} & 1/\gamma - 1 \end{pmatrix} \begin{pmatrix} \bar{u} & \bar{\rho} & 0 & 0 \\ 0 & \bar{\bar{u}} & 0 & 1/\bar{\rho} \\ 0 & 0 & \bar{\bar{u}} & 0 \\ 0 & \gamma\bar{p} & 0 & \bar{u} \end{pmatrix} \Delta\xi. \tag{6}$$

In the sequel, the first matrix in (6) is denoted by T_1. This matrix represents the transformation from conservative to primitive variables.

In a similar way the flux-difference Δg can be written as

$$\Delta g = T_1 \begin{pmatrix} \bar{v} & 0 & \bar{\rho} & 0 \\ 0 & \bar{\bar{v}} & 0 & 0 \\ 0 & 0 & \bar{\bar{v}} & 1/\bar{\rho} \\ 0 & 0 & \gamma\bar{p} & \bar{v} \end{pmatrix} \Delta\xi, \tag{7}$$

where

$$\bar{\rho}\,\bar{\bar{v}} = \overline{\rho v}. \tag{8}$$

Any linear combination of Δf and Δg can be written as

$$\Delta F = \alpha_1 \, \Delta f + \alpha_2 \, \Delta g = A \, \Delta \xi = T_1 \tilde{A} \, \Delta \xi, \tag{9}$$

where

$$\tilde{A} = \begin{pmatrix} \bar{w} & \alpha_1 \bar{\rho} & \alpha_2 \bar{\rho} & 0 \\ 0 & \bar{\bar{w}} & 0 & \alpha_1/\bar{\rho} \\ 0 & 0 & \bar{\bar{w}} & \alpha_2/\bar{\rho} \\ 0 & \alpha_1 \gamma \bar{p} & \alpha_2 \gamma \bar{p} & \bar{w} \end{pmatrix} \tag{10}$$

with

$$\begin{aligned} \bar{w} &= \alpha_1 \bar{u} + \alpha_2 \bar{v} \\ \bar{\bar{w}} &= \alpha_1 \bar{\bar{u}} + \alpha_2 \bar{\bar{v}}. \end{aligned} \tag{11}$$

For the case $\alpha_1^2 + \alpha_2^2 = 1$, the eigenvalues of the matrix $\tilde{A}$ are

$$\begin{aligned} \lambda_1 &= \bar{w} \\ \lambda_2 &= \bar{\bar{w}}, \end{aligned}$$

while λ_3 and λ_4 are given by

$$(\lambda - \bar{w})(\lambda - \bar{\bar{w}}) - \gamma \bar{p}/\bar{\rho} = 0. \tag{12}$$

With the definition of

$$\bar{c}^2 = \left(\frac{\bar{w} - \bar{\bar{w}}}{2} \right)^2 + \frac{\gamma \bar{p}}{\rho} \tag{13}$$

(12) can be written as

$$\left(\lambda - \frac{\bar{w} + \bar{\bar{w}}}{2} \right)^2 - \bar{c}^2 = 0$$

Hence

$$\lambda^{3,4} = \tilde{w} \pm \bar{c}, \tag{14}$$

where

$$\tilde{w} = \tfrac{1}{2}(\bar{w} + \bar{\bar{w}}). \tag{15}$$

The following Mach numbers can be defined:

$$\bar{M} = \bar{w}/\bar{c}, \qquad \bar{\bar{M}} = \bar{\bar{w}}/\bar{c}, \qquad \tilde{M} = \tilde{w}/\bar{c}. \tag{16}$$

The matrix $\tilde{A}$ can be normalized by

$$\tilde{A} = T_2 \bar{A} T_3 \tag{17}$$

where

$$T_2 = \text{diag}(\bar{\rho}\,\bar{c},\ \bar{c}^2,\ \bar{c}^2,\ \gamma\bar{p}\,\bar{c}) \tag{18}$$

$$T_3 = \text{diag}(1/\bar{\rho},\ 1/\bar{c},\ 1/\bar{c},\ 1/\gamma\bar{p}) \tag{19}$$

$$\bar{A} = \begin{pmatrix} \bar{M} & \alpha_1 & \alpha_2 & 0 \\ 0 & \bar{\bar{M}} & 0 & \alpha_1(1-\delta^2) \\ 0 & 0 & \bar{\bar{M}} & \alpha_2(1-\delta^2) \\ 0 & \alpha_1 & \alpha_2 & \bar{M} \end{pmatrix} \tag{20}$$

with $\delta = \frac{1}{2}(\bar{M} - \bar{\bar{M}})$.

The eigenvalues of the matrix $\bar{A}$ are

$$\lambda_1 = \bar{M}, \qquad \lambda_2 = \bar{\bar{M}}, \qquad \lambda_3 = \tilde{M} + 1, \qquad \lambda_4 = \tilde{M} - 1. \tag{21}$$

The left eigenvector matrix of $\bar{A}$ is

$$X = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & \alpha_1 & \alpha_2 & 1+\delta \\ 0 & -\alpha_2 & \alpha_1 & 0 \\ 0 & -\alpha_1 & -\alpha_2 & 1-\delta \end{pmatrix}. \tag{22}$$

The right eigenvector matrix of $\bar{A}$ is

$$X^{-1} = \begin{pmatrix} 1 & 0.5 & 0 & 0.5 \\ 0 & 0.5\alpha_1(1-\delta) & -\alpha_2 & -0.5\alpha_1(1+\delta) \\ 0 & 0.5\alpha_2(1-\delta) & \alpha_1 & -0.5\alpha_2(1+\delta) \\ 0 & 0.5 & 0 & 0.5 \end{pmatrix}. \tag{23}$$

Following the procedure of Steger and Warming for flux-vector splitting [1], the flux-difference given by (9) can be split into a positive and a negative part by

$$A^+ = X^{-1}\Lambda^+ X, \qquad A^- = X^{-1}\Lambda^- X \tag{24}$$

where

$$\Lambda^+ = \text{diag}(\lambda_1^+,\ \lambda_2^+,\ \lambda_3^+,\ \lambda_4^+)$$

$$\Lambda^- = \text{diag}(\lambda_1^-,\ \lambda_2^-,\ \lambda_3^-,\ \lambda_4^-)$$

with $\lambda_i^+ = \max(\lambda_i, 0)$, $\lambda_i^- = \min(\lambda_i, 0)$.

3. Construction of a Positive Set of Equations

Figure 1 shows a control volume with the node (i, j) located inside it. Also the nodes located inside the adjacent volumes are indicated. When a piecewise constant interpolation of variables is chosen, the flux through the surface $S_{i+1/2}$ of the control volume can be expressed by

$$F_{i+1/2}=\tfrac{1}{2}(F_i+F_{i+1}-|\Delta F_{i,i+1}|), \tag{25}$$

where F_i and F_{i+1} denote the fluxes computed with the values of the variables at the nodes (i, j) and $(i+1, j)$, respectively. For simplicity, in the above, the non-varying index is omitted.

Based on the previous section, clearly, a flux-difference between a flux calculated with values in the nodes (i, j) and $(i+1, j)$ can be written as

$$\Delta F_{i,i+1}=F_{i+1}-F_i=\Delta s_{i+1/2}A_{i,i+1}\,\Delta\xi_{i,i+1}, \tag{26}$$

where $\Delta s_{i+1/2}$ is the length of the surface $S_{i+1/2}$.

As shown in Fig. 1, the flux-difference across the surface $S_{i+1/2}$ can be written more explicitely as

$$\begin{aligned}\Delta F_{i,i+1}&=\Delta y_{i+1/2}\,\Delta f_{i,i+1}+\Delta x_{i+1/2}\,\Delta g_{i,i+1}\\&=\Delta s_{i+1/2}(\alpha_1\,\Delta f_{i,i+1}+\alpha_2\,\Delta g_{i,i+1}),\end{aligned} \tag{27}$$

where

$$\Delta s^2_{i+1/2}=\Delta x^2_{i+1/2}+\Delta y^2_{i+1/2}$$

$$\alpha_1=\Delta y_{i+1/2}/\Delta s_{i+1/2}, \qquad \alpha_2=\Delta x_{i+1/2}/\Delta s_{i+1/2}.$$

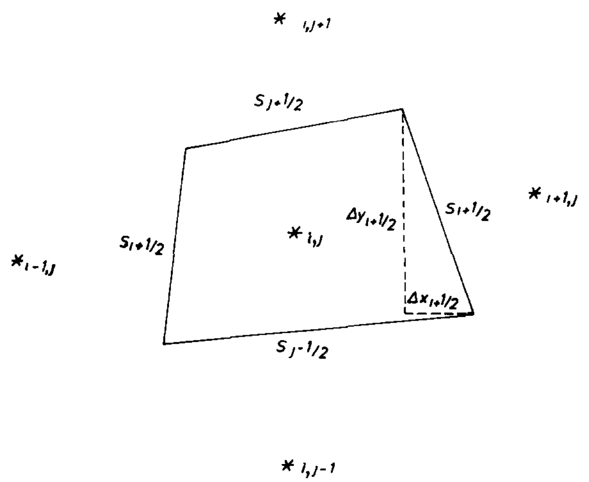

Fig 1 Control volume with piecewise constant interpolation

With the notation of the previous section, introduced in (9), (27) corresponds to (26).

Furthermore, the matrix $A_{i,i+1}$ can be split into a positive and a negative part:

$$A_{i,i+1} = A^+_{i,i+1} + A^-_{i,i+1}. \tag{28}$$

This allows the definition of the absolute value of the flux-difference by

$$|\Delta F_{i,i+1}| = \Delta s_{i+1/2}(A^+_{i,i+1} - A^-_{i,i+1})\, \Delta\xi_{i,i+1}. \tag{29}$$

With (26)–(29), the flux $F_{i+1/2}$ given by (25) can be written in either of the two following ways, which are completely equivalent:

$$\begin{aligned} F_{i+1/2} &= F_i + \tfrac{1}{2}\Delta F_{i,i+1} - \tfrac{1}{2}|\Delta F_{i,i+1}| \\ &= F_i + \Delta s_{i+1/2} A^-_{i,i+1}\, \Delta\xi_{i,i+1}, \end{aligned} \tag{30}$$

$$\begin{aligned} F_{i+1/2} &= F_{i+1} - \tfrac{1}{2}\Delta F_{i,i+1} - \tfrac{1}{2}|\Delta F_{i,i+1}| \\ &= F_{i+1} - \Delta s_{i+1/2} A^+_{i,i+1}\, \Delta\xi_{i,i+1}. \end{aligned} \tag{31}$$

The fluxes on the other surfaces of the control volume $S_{i-1/2}$, $S_{j+1/2}$, $S_{j-1/2}$, can be treated in a similar way as the flux on the surface $S_{i+1/2}$.

This leads to a flux-balance on the control volume indicated on Fig. 1, of the form

$$\begin{aligned} &\Delta s_{i+1/2} A^-_{i,i+1}(\xi_{i+1} - \xi_i) + \Delta s_{i-1/2} A^+_{i,i-1}(\xi_i - \xi_{i-1}) \\ &\quad + \Delta s_{j+1/2} A^-_{j,j+1}(\xi_{j+1} - \xi_j) + \Delta s_{j-1/2} A^+_{j,j-1}(\xi_j - \xi_{j-1}) = 0. \end{aligned} \tag{32}$$

The set of Eqs. (32) is both conservative and positive.

It is conservative since it exactly expresses the sum of fluxes on the control volume to be zero. It is positive since it can be put into the form

$$\begin{aligned} C\xi_{i,j} &= \Delta s_{i-1/2} A^+_{i,i-1}\xi_{i-1,j} + \Delta s_{i+1/2}(-A^-_{i,i+1})\,\xi_{i+1,j} \\ &\quad + \Delta s_{j-1/2} A^+_{j,j-1}\xi_{i,j-1} + \Delta s_{j+1/2}(-A^-_{j,j+1})\,\xi_{i,j+1}, \end{aligned} \tag{33}$$

where C is the sum of the matrix-coefficients in the right-hand side and where all matrix-coefficients involved have non-negative eigenvalues.

As a consequence of the positiveness, the set of equations of form (33) on all grid nodes can be solved by a vector variant of any scalar relaxation method. By a vector variant it is meant that in each node, all components of the vector of dependent variables ξ are relaxed simultaneously.

4. The Splitting Procedure in Steady Transonic Flow

In this and the following section, the flux-difference splitting is further detailed for a grid more or less aligned with the flow. As is clear from the previous sections, this restriction is not at all necessary. However, for an approximately aligned grid, some of the eigenvalues have fixed sign, simplifying the splitting considerably.

For the matrices $A_{i,i+1}$ and $A_{i,i-1}$ the Mach numbers involved in (21) are positive, but can be lower or higher than 1. For the matrices $A_{j,j+1}$ and $A_{j,j-1}$, the Mach numbers in (21) can be positive or negative, but their absolute value is less than 1.

For matrices of type $A_{i,i\pm 1}$, the positive and negative parts of the eigenvalues are

$$\lambda_i^+ = (\bar{M}, \bar{\bar{M}}, \tilde{M}+1, \tilde{M}^+) \tag{34}$$

$$\lambda_i^- = (0, 0, 0, \tilde{M}^- - 1), \tag{35}$$

where $\tilde{M}^- - 1 = \min(\tilde{M}-1, 0)$, $\tilde{M}^+ = \tilde{M} - \tilde{M}^-$.

With (34)–(35) and the definition of

$$\theta_1 = \tfrac{1}{2}\max(1-\tilde{M}, 0) \tag{36}$$

$$\theta_2 = 1 - \theta_1, \tag{37}$$

the split parts of the matrix $\tilde{A}$ defined in (10) become

$$\tilde{A}^+ = \begin{pmatrix} \bar{w} & \beta_{12}\bar{\rho} & \beta_{22}\bar{\rho} & \tau_1/\bar{c} \\ 0 & \bar{\bar{w}} + \alpha_{11}\bar{\bar{w}}_1 & \alpha_{12}\bar{\bar{w}}_1 & \beta_{12}/\bar{\rho} \\ 0 & \alpha_{12}\bar{\bar{w}}_1 & \bar{\bar{w}} + \alpha_{22}\bar{\bar{w}}_1 & \beta_{22}/\bar{\rho} \\ 0 & \beta_{12}\gamma\bar{p} & \beta_{22}\gamma\bar{p} & \bar{w} + \bar{w}_1 \end{pmatrix} \tag{38}$$

$$\tilde{A}^- = \begin{pmatrix} 0 & \beta_{11}\bar{\rho} & \beta_{21}\bar{\rho} & -\tau_1/\bar{c} \\ 0 & -\alpha_{11}\bar{\bar{w}}_1 & -\alpha_{12}\bar{\bar{w}}_1 & \beta_{11}/\bar{\rho} \\ 0 & -\alpha_{12}\bar{\bar{w}}_1 & -\alpha_{22}\bar{\bar{w}}_1 & \beta_{21}/\bar{\rho} \\ 0 & \beta_{11}\gamma\bar{p} & \beta_{21}\gamma\bar{p} & -\bar{w}_1 \end{pmatrix}, \tag{39}$$

where

$$\begin{aligned} &\alpha_{11} = \alpha_1^2 && \alpha_{12} = \alpha_1\alpha_2 && \alpha_{22} = \alpha_2^2 \\ &\beta_{11} = \alpha_1\theta_1 && \beta_{12} = \alpha_1\theta_2 && \\ &\beta_{21} = \alpha_2\theta_1 && \beta_{22} = \alpha_2\theta_2 && \\ &\bar{w}_1 = \theta_1(1-\delta)\,\bar{c} && \bar{\bar{w}}_1 = \theta_1(1+\delta)\,\bar{c} && \\ &\tau_1 = \theta_1/(1+\delta). && && \end{aligned} \tag{40}$$

For matrices of type $A_{j,j\pm 1}$, the positive and negative parts of the eigenvalues are

$$\lambda_i^+ = (\bar{M}^+, \bar{\bar{M}}^+, \tilde{M}+1, 0) \tag{41}$$

$$\lambda_i^- = (\bar{M}^-, \bar{\bar{M}}^-, 0, \tilde{M}-1), \tag{42}$$

where

$$\tilde{M}^+ = \max(\tilde{M}, 0) \qquad \tilde{M}^- = \min(\tilde{M}, 0)$$
$$\bar{\bar{M}}^+ = \max(\bar{\bar{M}}, 0) \qquad \bar{\bar{M}}^- = \min(\bar{\bar{M}}, 0).$$

With (41)–(42) and the definition of

$$\theta_1 = \tfrac{1}{2}(1 - \tilde{M}) \tag{43}$$

$$\theta_2 = 1 - \theta_1, \tag{44}$$

the split parts of the matrix $\tilde{A}$ defined in (10) become

$$\tilde{A}^+ = \begin{pmatrix} \bar{w}^+ & \beta_{12}\bar{\rho} & \beta_{22}\bar{\rho} & \tau_2/\bar{c} \\ 0 & \bar{\bar{w}}^+ + \alpha_{11}\bar{\bar{w}}_2 & \alpha_{12}\bar{\bar{w}}_2 & \beta_{12}/\bar{\rho} \\ 0 & \alpha_{12}\bar{\bar{w}}_2 & \bar{\bar{w}}^+ + \alpha_{22}\bar{\bar{w}}_2 & \beta_{22}/\bar{\rho} \\ 0 & \beta_{12}\gamma\bar{p} & \beta_{22}\gamma\bar{p} & 0.5\bar{w} + \bar{w}_2 \end{pmatrix} \tag{45}$$

$$\tilde{A}^- = \begin{pmatrix} \bar{w}^- & \beta_{11}\bar{\rho} & \beta_{21}\bar{\rho} & -\tau_2/\bar{c} \\ 0 & \bar{\bar{w}}^- - \alpha_{11}\bar{\bar{w}}_2 & -\alpha_{12}\bar{\bar{w}}_2 & \beta_{11}/\bar{\rho} \\ 0 & -\alpha_{12}\bar{\bar{w}}_2 & \bar{\bar{w}}^- - \alpha_{22}\bar{\bar{w}}_2 & \beta_{21}/\bar{\rho} \\ 0 & \beta_{11}\gamma\bar{p} & \beta_{21}\gamma\bar{p} & 0.5\bar{w} - \bar{w}_2 \end{pmatrix} \tag{46}$$

where

$$\begin{aligned} &\bar{w}_2 = 0.5(1 + \delta\tilde{M})\,\bar{c} && \bar{\bar{w}}_2 = 0.5(1 - |\bar{\bar{M}}| - \delta\tilde{M})\,\bar{c} \\ &\tau_2 = 0.5(1 - |\tilde{M}| + \delta\tilde{M})/(1 - \delta^2) && \\ &\bar{w}^+ = \max(\bar{w}, 0) && \bar{w}^- = \min(\bar{w}, 0) \\ &\bar{\bar{w}}^+ = \max(\bar{\bar{w}}, 0) && \bar{\bar{w}}^- = \min(\bar{\bar{w}}, 0). \end{aligned} \tag{47}$$

5. Boundary Conditions

At in- and outflow boundaries, due to an assumption of nearly uniform flow, the T_1-matrix is considered to be constant and the set of discrete equations simplifies to

$$\begin{aligned} &\Delta s_{i-1/2}\tilde{A}^+_{i,i-1}(\xi_i - \xi_{i-1}) + \Delta s_{i+1/2}\tilde{A}^-_{i,i+1}(\xi_{i+1} - \xi_i) \\ &\quad + \Delta s_{j-1/2}\tilde{A}^+_{j,j-1}(\xi_j - \xi_{j-1}) + \Delta s_{j+1/2}\tilde{A}^-_{j,j+1}(\xi_{j+1} - \xi_j) = 0, \end{aligned} \tag{48}$$

where the matrices $\tilde{A}^+_{i,i-1}$ and $\tilde{A}^-_{i,i+1}$, given by (38) and (39), further simplify, for subsonic in- and outflow, through

$$\theta_1 = \tfrac{1}{2}(1 - M), \qquad \theta_2 = \tfrac{1}{2}(1 + M)$$
$$\bar{w}_1 = \bar{\bar{w}}_1 = \theta_1 c = \tfrac{1}{2}c - \tfrac{1}{2}w \tag{49}$$
$$\delta = 0,$$

where M is the Mach number in the i-direction.

The set of Eqs. (48) contains contributions of one node outside the computational domain, either $i-1$ or $i+1$. Combinations of the equations in (48) are to be made, eliminating the outside node.

At the inflow boundary, one combination of Eqs. (48) is possible, eliminating the node $i-1$. It is easily seen that

$$d_1^T \tilde{A}^+ = 0$$

where

$$d_1^T = (0, \alpha_1/c, \alpha_2/c, \; -1/\gamma p).$$

The resulting equation is to be supplemented with three boundary conditions: stagnation temperature, stagnation pressure, and flow direction.

At the outflow boundary, three combinations of Eqs. (48) can be made by eliminating the node $i+1$. Clearly

$$d^T \tilde{A}^- = 0$$

for

$$d^T = d_2^T, d_3^T, \text{ or } d_4^T$$

where

$$d_2^T = (1/\rho, 0, 0, \; -1/\gamma p)$$
$$d_3^T = (0, \alpha_1/c, \alpha_2/c, 1/\gamma p)$$
$$d_4^T = (0, \; -\alpha_2, \alpha_1, 0).$$

The resulting equations are to be supplemented by one boundary condition. This can be the specification of the Mach number.

Figure 2 shows the choice of the control volume for a node on a solid boundary. The determination of the flux through the surfaces $S_{i+1/2}$, $S_{i-1/2}$, and $S_{j+1/2}$ is the same as for an internal node. In order to achieve consistency with the definition of the fluxes on the other surfaces, the flux on the surface S_j is to be defined by

$$F = F_{i,j} - \Delta s_j A_{i,j}^+ (\xi_{i,j} - \xi_{i,j-1}), \tag{50}$$

where $(i, j-1)$ is a fictitious node outside the flow field. Since the node (i, j) is on the surface S_j, the matrix A^+ in (50) is evaluated with the values of the variables in

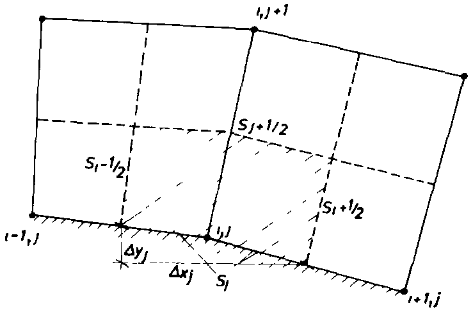

FIG 2 Control volume for a node on a solid boundary

this node. According to Fig. 2, the condition of impermeability on the surface S_j reads

$$\alpha_1 u_{i,j} + \alpha_2 v_{i,j} = 0. \tag{51}$$

A similar reasoning applies to a node on a northern boundary, involving the matrix $A^-_{i,j}$.

Due to the condition (51), the following simplifications can be made in $\tilde{A}^+$ and $\tilde{A}^-$, given by (45) and (46):

$$\begin{aligned} \bar{w} &= \bar{\bar{w}} = 0 \\ \delta &= 0 \\ \tau_2 &= 0.5 \\ \bar{w}_2 &= \bar{\bar{w}}_2 = 0.5c \\ \theta_1 &= \theta_2 = 0.5. \end{aligned} \tag{52}$$

By substitution of (52) into (45) and (46), it is easily seen that

$$d^T \tilde{A}^+ = 0$$

for

$$d^T = d_1^T, d_2^T, \text{ or } d_3^T,$$

where

$$\begin{aligned} d_1^T &= (\gamma p/\rho, 0, 0, -1) \\ d_2^T &= (0, \alpha_2, -\alpha_1, 0) \\ d_3^T &= (-c/\rho, \alpha_1, \alpha_2, 0). \end{aligned}$$

Similarly,

$$d^T = d_1^T,\ d_2^T,\ \text{or}\ d_4^T$$

with

$$d_4^T = (+c/\rho,\ \alpha_1,\ \alpha_2,\ 0).$$

Using (9) and (51) it is seen that

$$e^T A^+ = 0$$

for

$$e^T = e_1^T,\ e_2^T,\ e_3^T$$

where

$$e_1^T = (H, 0, 0, -1)$$
$$e_2^T = (-(u^2+v^2), u, v, 0)$$
$$e_3^T = (-c, \alpha_1, \alpha_2, 0)$$

and

$$e^T A^- = 0$$

for

$$e^T = e_1^T,\ e_2^T,\ e_4^T,$$

where

$$e_4^T = (+c, \alpha_1, \alpha_2, 0).$$

The nodal equations at a solid boundary are premultiplied by e_i^T, with $i = 1, 2, 3$ on a southern boundary and $i = 1, 2, 4$ on a northern boundary, leading to three significant equations. These are supplemented with the kinematic boundary condition (51).

6. Numerical Example

Figure 3 shows the well known GAMM-test case [12] for transonic flows, discretized by a grid with 36×12 elements. In the actual computation a once more refined grid was used with 72×24 elements. Vertex-based finite volumes, as indicated in Fig. 2, were used.

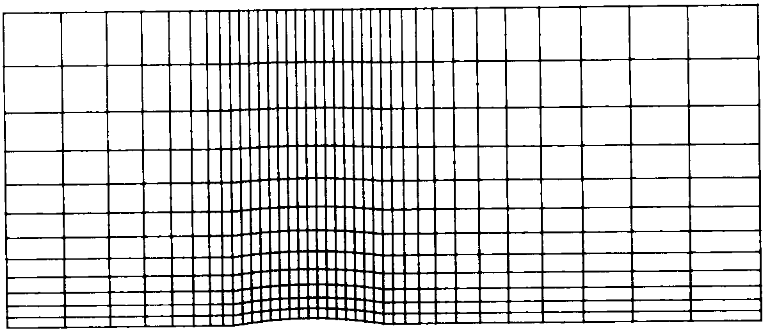

FIG 3 Coarse computational grid

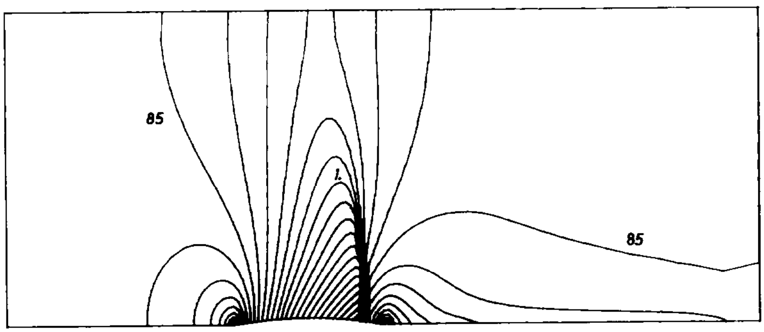

FIG 4 Iso-Mach lines for the geometry of Fig 3 on a once more refined grid

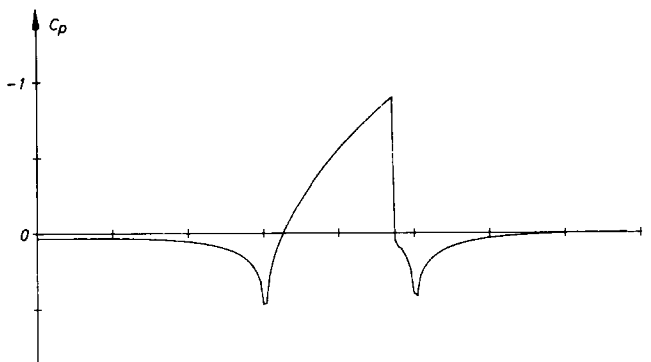

FIG. 5 Calculated pressure distribution on the southern boundary

At inflow, the specification of a horizontal flow direction was used as a boundary condition. At outflow the Mach number was fixed at 0.85.

Starting from a uniform flow at Mach number 0.85, the discrete equations were solved by Jacobi relaxation with a relaxation factor 0.95. It is to be remarked that for systems of equations, the theoretical maximum relaxation factor is not 2, but 1. It took about 300 iterations to obtain a solution in which the nodal Mach numbers are within $1\,^0/_{00}$ of their fully converged values. The computation time for these 300 iterations is about 7.5 cpus on the cyber-205 with a vectorized version of the code.

Figure 4 shows the iso-Mach lines for the fully converged solution plotted by piecewise linear interpolation within the elements of the grid. Figure 5 shows the surface pressure distribution on the southern boundary.

The obtained solution coincides almost with the solution obtained from the most reliable time-marching methods reported in [12]. However, unlike most time-marching solutions, due to the guaranteed positiveness everywhere, the solution has no wiggles in the shock region.

7. Conclusion

It was shown that by the polynomial flux-difference approach a conservative discretization for steady Euler equations can be obtained, which is of positive type. The solution of the equations can be obtained with relaxation methods, leading to elliptic solution procedures for this hybrid partial differential problem.

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